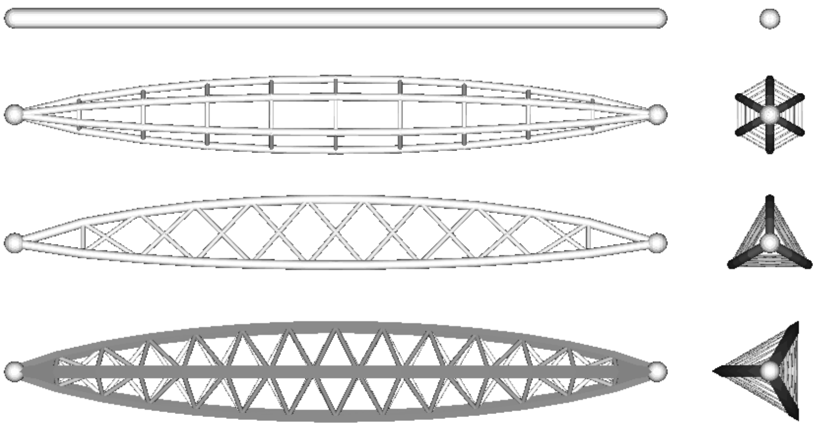
Competition Struts



Strut no.	Score
0	1.0
1	3.1
2	13.3
3	16.3

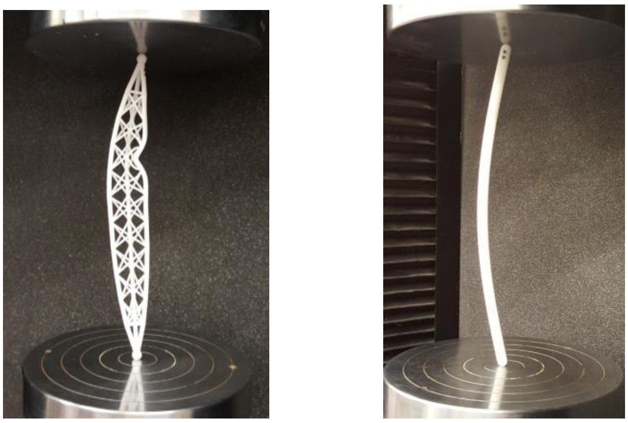
Figure 1: Competition struts and their scores

Competition struts, like those shown in Fig 1, are so called because each has a score, a number that is a measure of its performance. Strut 0 is a solid cylindrical rod and, being the strut with no internal shape but having the same length and mass as the other three, and, being made of the same material, it is the basis for measuring the others' performance. It is called their *reference strut*.

$$\text{A competition strut's score} = \frac{\text{strength of the competition strut}}{\text{strength of its reference strut}}$$

The shaping of its material can confer advantage to a strut. Deploying material away from the strut's axis increases its bending stiffness and hence its buckling strength. However, the act of separating material from itself introduces ligaments which themselves are liable to buckle, so an assessment of buckling strength must consider both global and local buckling modes, which is why optimal shapes require careful design.

A competition strut should fail by elastic buckling, not by material overstress; the material used should be strong enough not to fail prior to the strut's elastic buckling load being reached. Elastic buckling is most easily explored numerically. Computer analysis software can predict the elastic buckling strength of a numerically modelled strut. The reference strut does not need numerical modelling as its strength can be calculated accurately using Euler's well-known buckling formula, $P = \pi^2 EI/L^2$.



Strength = 450 N   Strength = 35.2 N

Figure 2: Testing of Struts 2 and 0

Alternatively, the score can be derived by measuring the strengths of physical models, using a testing machine like that shown in Fig 2. Physical struts should have spherical ends, each having the diameter of the reference strut, to allow them to roll on the platens without picking up end moments, thus ensuring that the struts behave as if they are pinned. The effective length of each strut is the distance between the centres of its two end spheres.

The tapered profile

A tapered profile, like that of Fig 3, may be defined by a *profile function*, p, where

$$y = b\mathrm{p}(x/a)$$

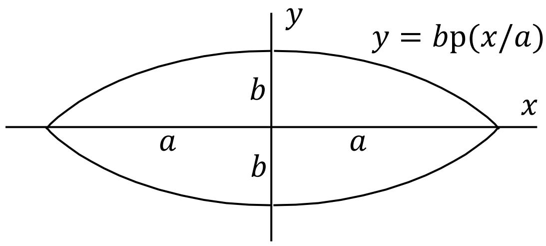


Figure 3: Tapered profile

A feature of an optimally light structure is that under its design load its stress distribution is uniform [1]. It is therefore expected that the cross-sectional area of a competition strut will, if possible, be uniform along its length. A tapered lattice member is therefore expected to have curved chord members which have uniform sectional area along their lengths, and a tapered tube is expected to have varying thickness, this being a minimum at its central section and increasing towards each end where the section becomes solid, thus maintaining a constant sectional area throughout its length.

The strength of a strut, when it buckles by bowing, is proportional to its bending stiffness. This stiffness will be optimal for a bowing deflection which causes the bending stresses to be uniformly spread down the length of the member. This property belongs to the profile defined by

$$y\frac{d^2y}{dx^2} = -\frac{\pi}{2}\left(\frac{b}{a}\right)^2$$

Its shape, shown in Fig 3, is defined numerically in Table 1. The shape and its properties are described in [2].

Table 1: Tabulated values for one quadrant of the tapered profile

α	$\mathrm{p}(\alpha)$	α	$\mathrm{p}(\alpha)$	α	$\mathrm{p}(\alpha)$	α	$\mathrm{p}(\alpha)$
0	1.00	0.3170	0.92	0.6017	0.70	0.8793	0.30
0.1127	0.99	0.3538	0.90	0.6467	0.65	0.9272	0.20
0.1593	0.98	0.3869	0.88	0.6879	0.60	0.9681	0.10
0.1950	0.97	0.4314	0.85	0.7258	0.55	0.9856	0.05
0.2249	0.96	0.4959	0.80	0.7610	0.50	0.9976	0.01
0.2750	0.94	0.5519	0.75	0.8242	0.40	1.0000	0.00

The bowing buckling mode shape also follows this profile, and the buckling load is

$$P = 2\pi EI_0/L^2$$

where I_0 is the inertia at its middle section and the radius of gyration at any section varies according to this same profile, i.e. $r = r_0\mathrm{p}(x/a)$, and the sectional area, A, is constant along the length, so at any section $I = Ar^2$. The similarity of its buckling formula to Euler's is striking.

The arc length of one quadrant of this profile $= a\left(1 + \frac{\pi}{4}\left(\frac{b}{a}\right)^2\right)$ for small values of b/a.

The area within the whole profile shown in Fig 3 $= 2\sqrt{2}ab$, which lies between πab, the area within the circumscribing ellipse, and $2\frac{2}{3}ab$ for the area bounded by two opposing parabolic arcs.

Local shear buckling

Another peculiar feature of the profile, p, is that its member's buckling strength of $2\pi EI_0/L^2$ in the bowing mode is unaffected by the member's shear flexibility, i.e. it is unaffected by its bracing. This is because, when a strut with this profile bends in this mode, there is no shear transfer between the chords. However, all such members rely totally on their shear stiffness to avoid the local shear buckling mode shown in Fig 4. The bracing of latticed struts must be designed to avoid this buckling mode.

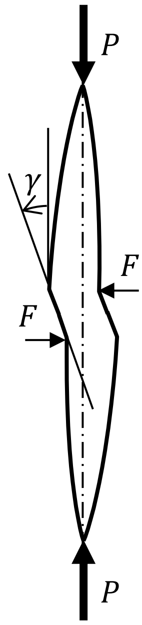


Figure 4: Shear buckling

A perturbation causes the central zone to shear by an angle, γ. There, an axial load, P, has a shear component of γP. If the shear stiffness of the central zone is K_S, expressed as a shear force per unit shear strain, then the shear force in the central zone is $K_S\gamma$. But the shear force there is $F + \gamma P$. Equating the two:

$$K_S\gamma = F + \gamma P$$

Buckling occurs when the force, P, on its own, is enough to cause the perturbation, i.e. when $F = 0$ and $P = K_S$. So, K_S is the buckling strength.

Strut 1 of Fig 1 fails by buckling in this mode.

Tube versus lattice

First, consider a tube which is L long, which has a middle surface diameter of D and a thickness there of t. Its sectional area is πDt throughout its length. So, if the diameter of its reference strut is d_{ref}, then $\pi {d_{ref}}^2/4 = \pi Dt$ and $d_{ref} = 2\sqrt{Dt}$. The two radii of gyration are $D/2\sqrt{2}$ and $d_{ref}/4$, from which the score, the ratio of the two buckling strengths, can be calculated as

$$\text{Score for a tube} = \frac{2\pi EI_0/L^2}{\pi^2 EI_{ref}/L^2} = \frac{2I_0}{\pi I_{ref}} = \frac{D^2/4}{\pi {d_{ref}}^2/16} = \frac{4}{\pi}\frac{D^2}{{d_{ref}}^2} = \frac{1}{\pi}\frac{D}{t}$$

The struts of Fig 1 are all 200 mm long with a minimum thickness of 1.5 mm. If these same constraints were put on a tube, then a 75 mm diameter tube would have a score of about 16 according to the formula. Lattices under these dimensional constraints can achieve scores much higher than this.

Next, consider a lattice which is L long and has m chords, each of diameter, d, each divided by its bracing into n equal segments, as shown in Fig 5. When the chord segments are slender, little bracing is required, so the mass of the bracing is ignored for this approximate assessment.

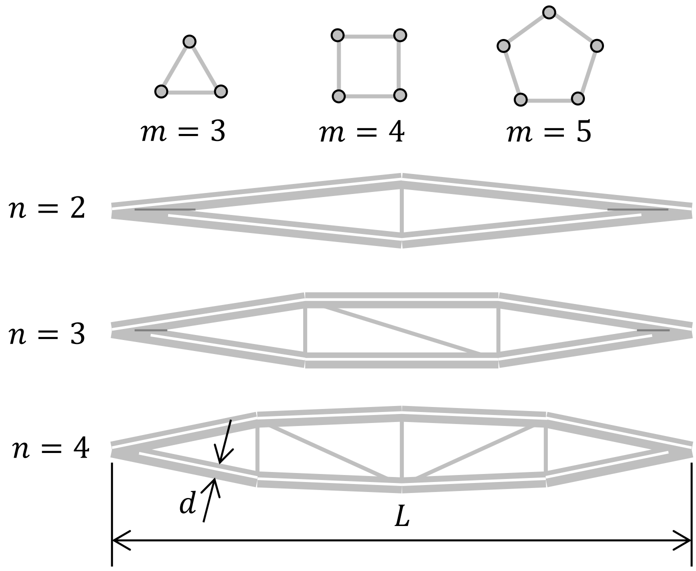


Figure 5: Truss configurations

Sectional area $= m\frac{\pi d^2}{4}$

If the diameter of the reference strut is d_{ref}, then $\frac{\pi {d_{ref}}^2}{4} = m\frac{\pi d^2}{4}$, and $d_{ref} = \sqrt{m}d$.

The buckling strength of the reference strut, $P_{ref} = \pi^2 E \frac{\pi {d_{ref}}^4}{64} \frac{1}{L^2} = \pi^2 E \frac{\pi m^2 d^4}{64} \frac{1}{L^2}$.

If these struts are stocky enough not to fail by global buckling, they will fail when the chords buckle, when the strength of the competition strut, $P_{comp} = m\pi^2 E \frac{\pi d^4}{64} \frac{n^2}{L^2}$.

Hence, the score $= \frac{P_{comp}}{P_{ref}} = \frac{n^2}{m}$

Therefore, to maximize the score, the number of chords should never be more than 3 and a high scoring strut should be expected to have many chord segments. This approximate formula scores the three struts of Fig 1 as 17, 40 and 65, all of them are too high, mainly because the mass of the bracing has been ignored.

When n is large, bracing contributes significantly to a lattice's mass, so a different calculation is required. A 3-chorded lattice, like Strut 2 of Fig 1, is investigated, each of its chords having a diameter, d, and divided into n equal segments. The chords are cross braced using members that are t in diameter. The pitch circle diameter of the chords in the central zone is D.

In the central zone the centrelines of the chords are $\frac{\sqrt{3}}{2}D$ apart. The average depth of the profile of Fig 3 is $\frac{1}{\sqrt{2}}a$, so, for the volume calculation, it will be assumed that the bracing spans an average distance of $\frac{\sqrt{3}}{2\sqrt{2}}D$. This makes the average length of each brace $\sqrt{\frac{3}{8}D^2 + L^2/n^2}$. There are $3(n-2)$ sets of cross bracing.

Volume of bracing $V_B = 6(n-2)\pi \frac{t^2}{4} \sqrt{\frac{3}{8}D^2 + L^2/n^2}$

The length of each chord $= L\left(1 + \frac{\pi}{4}\left(\frac{D}{L}\right)^2\right)$

Volume of the chords $V_C = 3\pi \frac{d^2}{4} L\left(1 + \frac{\pi}{4}\left(\frac{D}{L}\right)^2\right)$

Volume of the reference strut $= V_B + V_C$, from which ${d_{ref}}^2 = \frac{4}{\pi}(V_B + V_C)/L$

The buckling strength of the reference strut $P_{ref} = \pi^2 E \frac{\pi {d_{ref}}^4}{64} \frac{1}{L^2} = \pi E \frac{(V_B+V_C)^2}{4} \frac{1}{L^4}$

The competition strut may fail in three different ways: by global buckling, by local chord buckling and by local shear buckling. An optimum design may be expected to fail in all three modes simultaneously if there are enough independent design parameters.

The global buckling strength of the strut, $P_G = 2\pi E \left(3\pi \frac{d^2}{4}\right) \frac{D^2}{8} \frac{1}{L^2} = 1.85E \frac{d^2 D^2}{L^2}$

Local chord buckling results in a rippling of the chord, the ripples having a half wavelength equal to the supported length of each chord segment. Adjacent chord segments give no rotational restraint because they too are buckling, but the bracing contributes enough restraint for the effective buckling length of the chord to be about 0.9 times the chord's segment length. This factor of 0.9 is assumed in this calculation.

The local chord buckling strength of the strut, $P_C = 3\pi^2 E \left(\pi \frac{d^4}{64}\right) \frac{n^2}{0.9^2 L^2} = 1.79E \frac{n^2 d^4}{L^2}$

When a shear force, F, is applied to a panel of cross bracing, which is l long and h high, and shears by an angle, γ, as shown in Fig 5, one brace shortens while the other lengthens by the same amount. The change in length is $\gamma l \sin\theta$, which produces an axial strain in each brace of $\gamma l \sin\theta\,/(l/\cos\theta)$, i.e. $\gamma\sin\theta\cos\theta$, where θ, the bracing angle, $= \tan^{-1} h/l$. The resulting force in each brace is $(\gamma\sin\theta\cos\theta)E\pi t^2/4$, and the shear is

$$F = \tfrac{1}{2}\gamma E\pi t^2 \sin^2\theta\cos\theta$$

The shear stiffness of each panel is $K_P = F/\gamma$

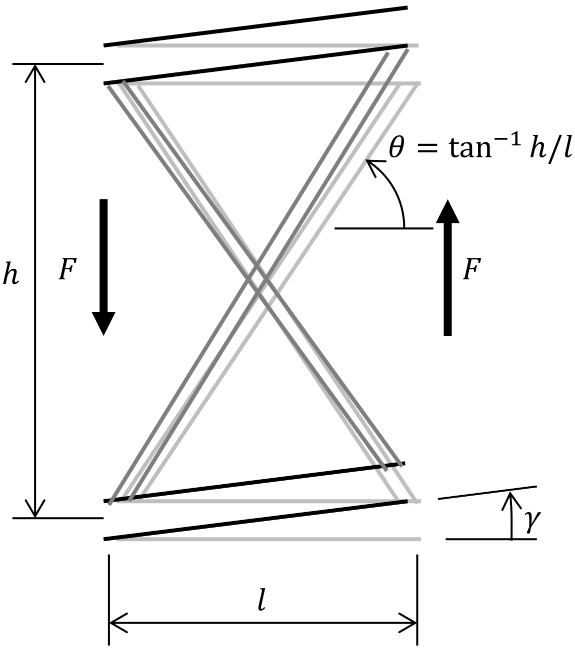


Figure 5: Shear deformation

The competition strut has 3 such panels. Suppose the shear strain in the member acts at an angle ϕ to one of the panels, as shown in Fig 6. Then the shears in the three panels are as shown in the figure and the shear forces in the same direction as the shear strain, γ, is

$$K_P\gamma(\cos^2\phi + \cos^2(60+\phi) + \cos^2(60-\phi)) = 1.5K_P\gamma \text{ whatever the value of } \phi$$

Hence, the shear stiffness of the whole section, $K_S = 1.5K_P = \tfrac{3}{4}E\pi t^2\sin^2\theta\cos\theta$

which is the local shear buckling strength, i.e. $P_S = K_S = \tfrac{3}{4}E\pi t^2\sin^2\theta\cos\theta$

where $\theta = \tan^{-1}\frac{h}{l} = \tan^{-1}\frac{\sqrt{3}}{2}\frac{nD}{L}$ in the central zone, where the buckling will occur.

Summary:

The strength of the competition strut P_{comp} is the least of the values of P_G, P_C and P_S, where

$$P_G = 1.85E\frac{d^2D^2}{L^2} \qquad (1)$$

$$P_C = 2.27E\frac{n^2d^4}{L^2} \qquad (2)$$

$$P_S = 2.36Et^2\sin^2\theta\cos\theta \quad \text{where } \theta = \tan^{-1} 0.866\frac{nD}{L} \qquad (3)$$

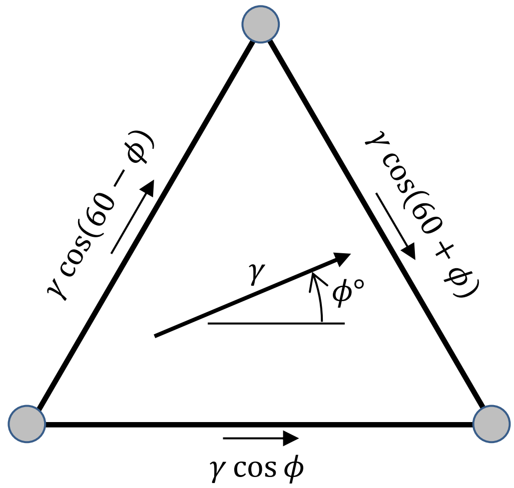


Figure 6: Shear strain distribution

The strength of the reference strut is

$$P_{ref} = 0.785E\frac{(V_B+V_C)^2}{L^4}$$

$$\text{where } V_B = 1.5(n-2)\pi t^2\sqrt{0.375D^2 + \frac{L^2}{n^2}}$$

$$\text{and } V_C = 2.36Ld^2\left(1 + \frac{0.785D^2}{L^2}\right) \qquad (4)$$

The score is P_{comp}/P_{ref} (5)

Testing this out on Strut 2, for which:

$E = 1650$ N/mm^2
$L = 200$ mm

$n = 11$
$D = 27.18$ mm
$d = 3$ mm
$t = 1.5$ mm

	$P_G = 1.85E\frac{d^2D^2}{L^2}$	507 N
	$P_C = 1.79E\frac{n^2d^4}{L^2}$	724 N
	$P_S = 2.36Et^2\sin^2\theta\cos\theta$	3350 N
where	$\theta = \tan^{-1} 0.866\frac{nD}{L}$	52.3°
	$V_B = 1.5(n-2)\pi t^2\sqrt{0.375D^2 + \frac{L^2}{n^2}}$	2352 mm^2
	$V_C = 2.36Ld^2\left(1 + \frac{0.785D^2}{L^2}\right)$	4310 mm^2
	$P_{ref} = 0.785E\frac{(V_B+V_C)^2}{L^4}$	35.9

Score $= \frac{471}{35.4} = 14.1$

These results agree with the design intentions for Strut 2: it was designed to fail by global buckling and to have a safety factor of 1.5 against chord buckling, i.e. the factor $\frac{724}{507} = 1.43$. The reason for this safety factor was to preserve the struts during testing. One test specimen suffered chord buckling, the one shown in Fig 2, one fractured at one end, and the remaining three were undamaged by the test because they were unloaded quickly as soon as they had passed maximum load. Another constraint on the design was to match it to a 6.5 mm diameter reference strut, i.e. to have a volume of 6640 mm^3. The bracing diameter of 1.5 mm was a constraint, but not the chord diameter, but the roundness of its 3 mm dimension appealed and the pitch circle diameter of 27.18 mm, while not optimum, was considered satisfactory.

Optimal latticed struts with solid circular sections

Once the structural performance of a latticed strut with this topology is described by the equations 1 to 4, finding the shape that defines the strut with the maximum score can be determined numerically. The unknowns are E, L, n, D, d and t. Since the strengths of the two struts are both proportional to their material's stiffness, E, this can be ignored. Both L and t have been specified as 200 mm and 1.5 mm, respectively, leaving 3 variables, n, D and d. Fig 7 shows contours of the score as D and d vary, with $n = 11$. Considering the contours as height, the plot represents the apex of a 3-faced pyramid. Each face represents one of the three buckling modes. The point representing Strut 2 is plotted in the lower left-hand corner, having a chord diameter of 3 mm and a pitch circle diameter of 27.18 mm. It fails by global buckling. The edges where the surfaces meet, shown as dashed lines, indicate those designs that fail simultaneously in two modes. Strut 2 was deliberately designed to keep some way clear of this line. Its score of 14.1 is confirmed by the contours.

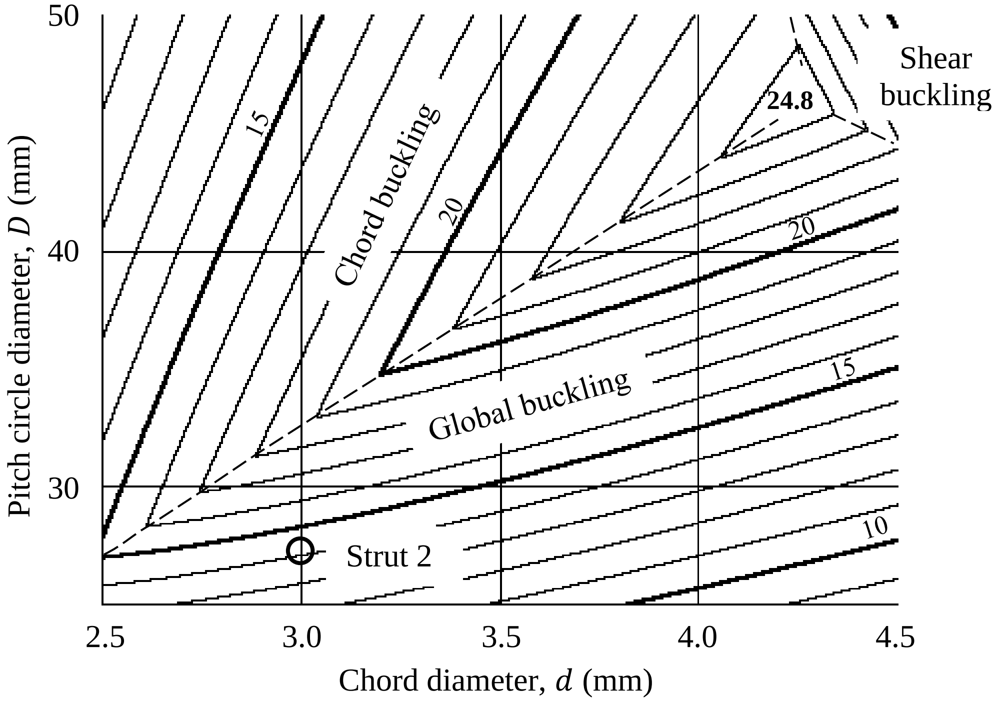


Figure 7: Contour plot of score versus the two diameters, with $n = 11$

The optimum design with $n = 11$, at least the one with the greatest score, is the one which fails in all three modes at once, represented by the point at the apex of the pyramid, scoring 24.8.

But is $n = 11$ optimal? Investigation shows that $n = 14$ gives the highest possible score for this typology, scoring 26.4, with $D = 49.8$ mm and $d = 3.62$ mm. Its strength is 2480 N, with $E = 1650$ N/mm^2. The volume of the strut is 10.77 cc, with 60.2% of the material being in the chords. The diameter of its reference strut is 8.28 mm, with a buckling strength of 93.9 N.

It is a feature of competition struts that thinner material results in higher scores. Fig 9 shows the scores when the bracing diameter is reduced to 1 mm. The highest possible score is obtained when $n = 17$, and the scores in Fig 9 are for this bracing configuration. The maximum score has increased to 42.4, with $D = 44.2$ mm and $d = 2.64$ mm. With $E = 1650$, its strength is 1040 N, less than half that of the previous strut. The volume of the strut is 5.50 cc, with 62.1% of the material being in the chords. The diameter of its reference strut is 5.92 mm, with a buckling strength of 24.5 N.

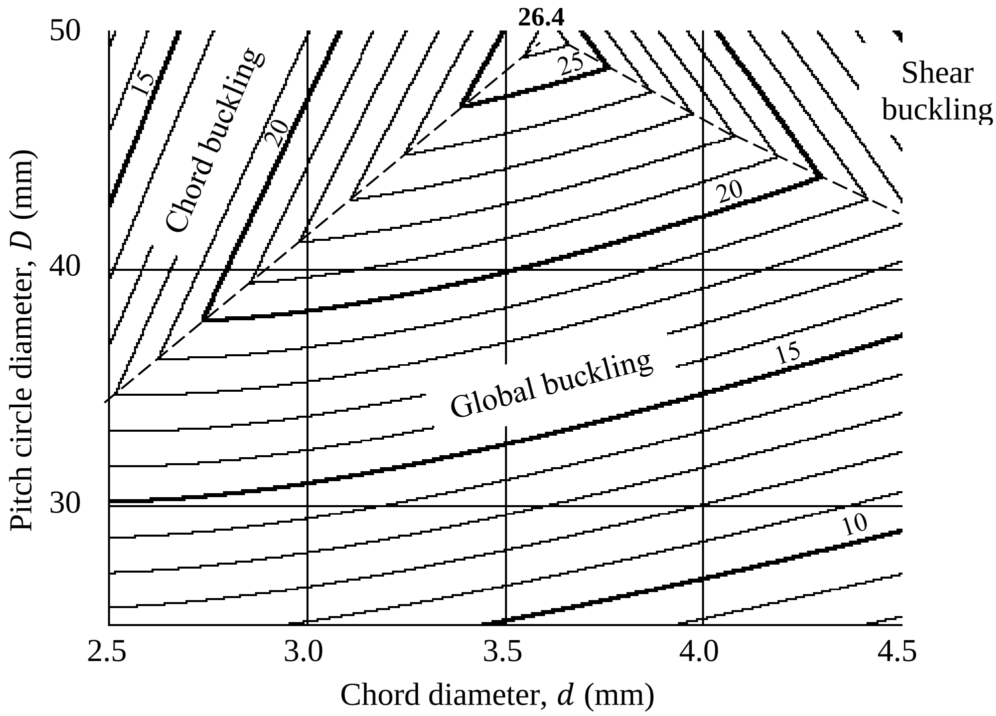


Figure 8: Scores for 1.5 mm dia. bracing, with $n = 14$, the optimum value

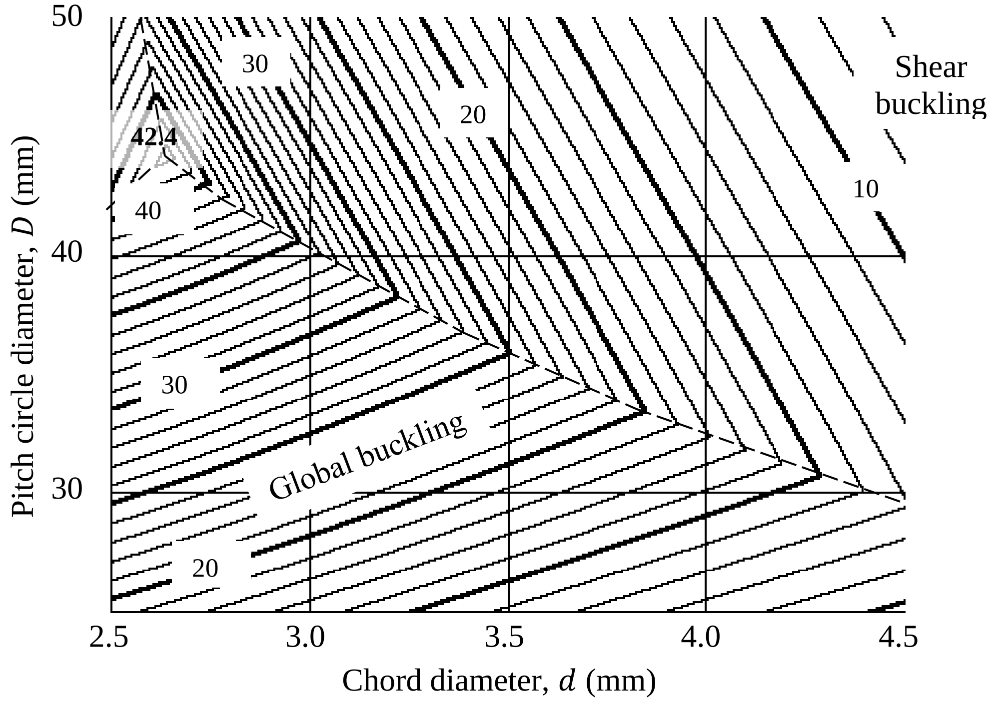


Figure 9: Scores with 1 mm dia. bracing and the optimum $n = 17$

These optimal struts, which buckle in three modes at once, are not just optimal for score but are optimal for specific strength, i.e. strength/volume. Figure 10 shows a contour plot of specific strength for the set of struts whose performance is illustrated in Fig 9. The optimum strut is the same, with $D = 44.2$ mm and $d = 2.65$. Properties of optimal struts having the topology of Strut 2 are given in Table 2. These struts give optimal values of both specific strength and score but notice that, as the score increases, the specific strength decreases. This does not imply that low scoring struts are better. The dimensional constraints imposed on a lightweight strut adversely affect a strut's specific strength.

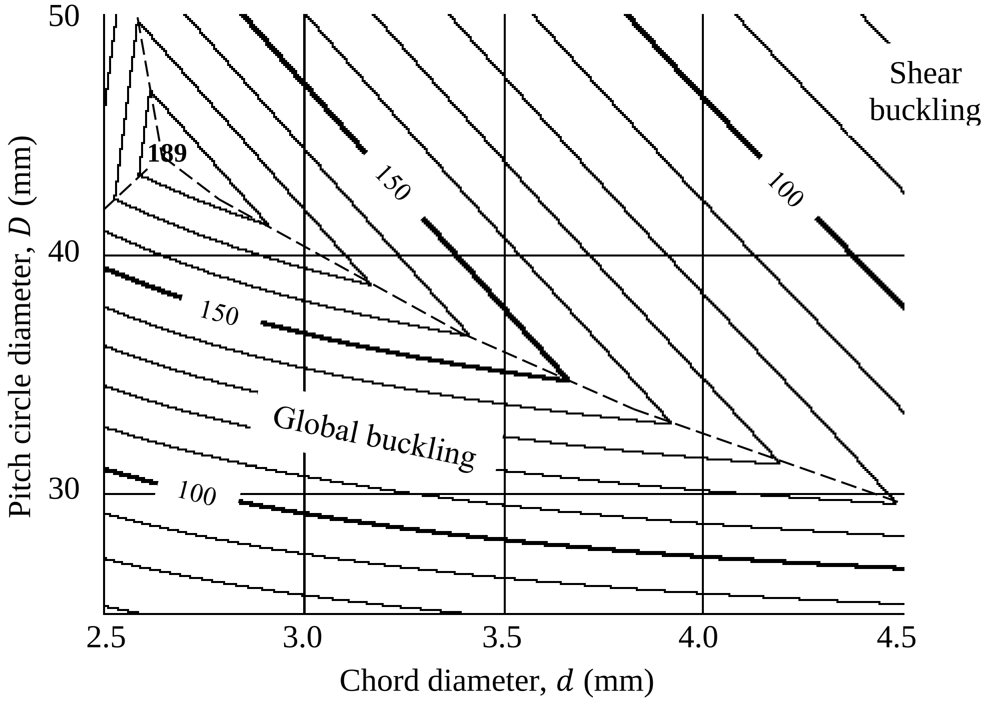


Figure 10: Specific strength (N/cc) with 1 mm dia bracing and $n = 17$

Table 2: Optimal 'Strut 2' performance with decreasing brace diameter

Brace diameter t (mm)	No. of segments n	Pitch circle dia. D (mm)	Chord diameter d (mm)	Volume V (cc)	Strength P (N)	Specific strength P/V (N/cc)	Score
2	11	52.6	4.86	18.0	4980	276	18.9
1.75	12	50.9	4.32	14.3	3690	257	22.1
1.5	13	48.8	3.81	11.1	2640	238	26.5
1.25	15	47.0	3.19	8.03	1710	213	32.7
1	17	44.2	2.65	5.52	1044	189	42.4
0.75	20	40.9	2.08	3.39	551	162	59.1
0.5	24	36.2	1.53	1.75	234	134	94.4

Notice that reducing the minimum material thickness, t, increases the score: indeed, halving the thickness doubles the score over the range investigated, so it is logical to investigate the effect of shaping the chords into more complicated sections, with a minimum material thickness of t.

Optimal latticed struts with optimally shaped sections

Table 2 shows that the score is roughly inversely proportional to the minimum thickness of its strut's material, and since the material thickness in the chords is always much greater than that in the bracing, it must be questioned how scores could be increased further by shaping the chords to have the same minimum thickness of material as the bracing. To discover the answer to this question just one new variable is introduced.

Any chord with cross-sectional area, A, symmetrical, with one radius of gyration, r, can be substituted by a solid circular member made of a different material, while remaining structurally equivalent. If this substitute member has cross-sectional area, A', and radius of gyration, r', for it to be structurally equivalent this substitute member must have the same axial and bending stiffness as the one it is representing:

i.e. $E'A' = EA$

and $E'A'r'^2 = EAr^2$

Eliminating $E'A'$ from these two equations, $r' = r$. So, the substitute member must have the same radius of gyration as that of the member it is representing.

If the substitute member's diameter $= d$, then its sectional area is $A' = \pi d^2/4$ and its radius of gyration, $r' = d/4$. Hence $d = 4r$, i.e. the diameter of the substitute member is four times the radius of gyration of the member it is representing.

The substitute material is less stiff than the actual material. This stiffness factor is

$$i = \frac{E}{E'} = \frac{A'}{A} = \frac{\pi d^2}{4A} = \frac{4\pi r^2}{A}$$

and $E' = E/i$.

So, the member having a sectional area, A, and radius of gyration, r, can be represented by a solid member of diameter $d = 4r$ made of a material which has a stiffness of E/i. The factor, i, is the new variable which frees the chords from being restricted to solid circular sections. It is not independent because the section to replace the solid circular section is subject to a minimum material thickness constraint of t. Each solid section has its own relative stiffness factor, i: i is a section property. Since the objective is to get as far as possible from a solid circular section, which has $i = 1$, it is proposed to select the section which has the greatest value for i.

Since material thickness is such an important parameter, it is worth considering how this should be defined. Fig 11 shows three solid sections, one circular and two equilateral-triangular, all three posing as having a thickness of t. It is suggested that the section on the right is an impostor because the thickness of an equilateral-triangular section is not its side length, but its height, this being the shortest distance between opposite parallel surfaces.

The algorithm that drove the Strut 2 optimization had d and t as parameters. The only adjustment that is required for the shaping of the chord section is to prescribe i as a function of d and t, then to continue as if the chords were still solid but whose material stiffness is E/i.

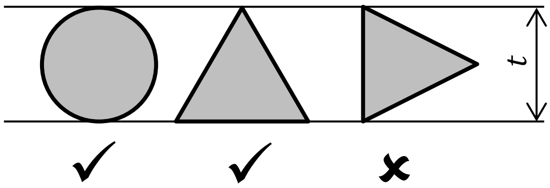


Figure 11: The definition of thickness

For solid circular sections, $i = 1$. For all other sections, it is greater than 1. For solid square sections, $i = 1.05$, and for a solid equilateral triangular section, $i = 1.21$. Sections that are the most suitable for the chords of a competition strut are those with the largest value of i, and these are shown in Fig 12. The smallest possible chord is a solid triangular section with a depth of t, a side length of $2t/\sqrt{3}$, and a radius of gyration of $t/3\sqrt{2}$, which can be represented by a solid circular section with a diameter which is four times this, i.e. $d = 2\sqrt{2}t/3 = 0.943t$. This is the smallest permissible chord diameter, yes, smaller than the permissible thickness, because this diameter is not a thickness, it is the diameter of a substitute member. Increasing the value of d up to $2.16t$ does not alter what shape of section is the most efficient, i.e. has the greatest value of i: it is still the solid triangular section. As d increases further, the triform section begins to give the greatest possible i value. This is like a cruciform section, but with three fins instead of four.

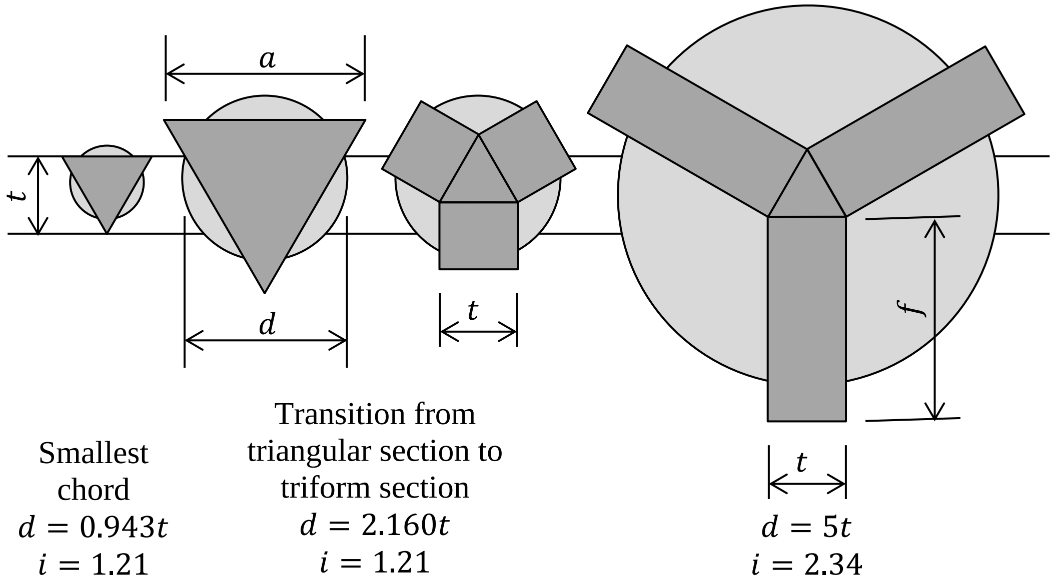


Figure 12: Suitable chord sections with their solid circular substitutes

Fig 13 shows how the relative stiffness factor, i, varies with d/t. With zero length fins, with $f = 0$, the triform section is triangular, then, as the fins grow, the shape becomes more rounded, which causes i to decrease, but when the fins are long enough its i-values exceed that of the solid triangle.

Table 3 gives the properties of some triform sections. The quadratic curve passing through the point (2.160, 1.209) with the best least squares fit to the last four rows of the table is

$$i = 0.488 + 0.306\left(\frac{d}{t}\right) + 0.0127\left(\frac{d}{t}\right)^2 \qquad \text{when} \quad \frac{d}{t} > 2.160$$

For smaller values of d/t

$$i = 1.209 \qquad \text{when} \quad 0.943 < \frac{d}{t} < 2.160$$

For consistency, the braces will be triangular in section with a thickness of t. They will be considered as solid circular sections having a diameter of $0.943t$ and a stiffness of $E/1.209$.

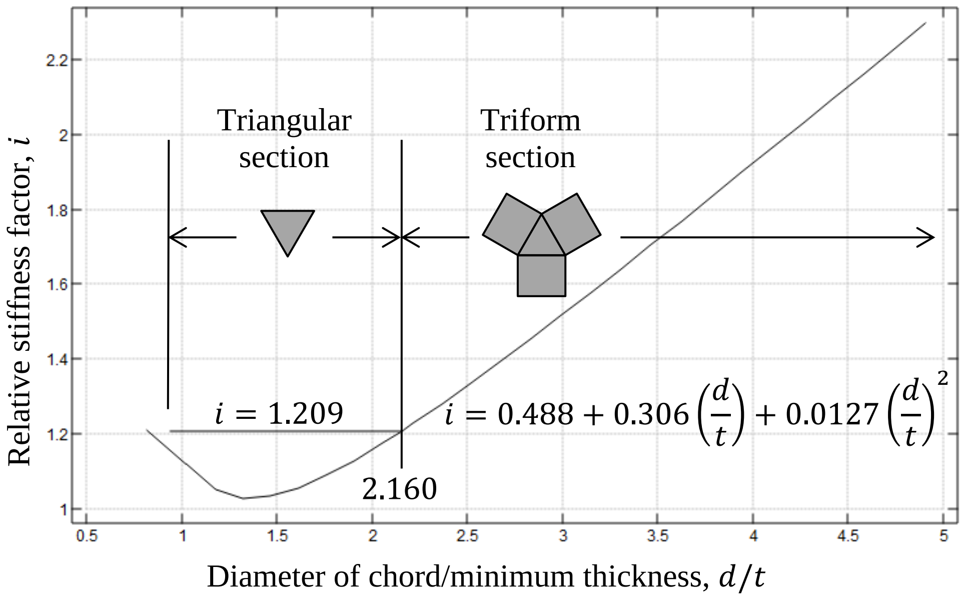


Figure 13: Optimal variation of i with d/t

Table 3: Dimensions of sections that maximise the relative stiffness factor, i

Equivalent circular section, diameter = d d/t	Triangular section with side length = a a/t	Triform section with fins, f long f/t	Relative stiffness factor i
0.943	1.155	-	1.209
2.160	2.646	-	1.209
2.160	-	0.866	1.209
2.5	-	1.087	1.329
3	-	1.406	1.519
4	-	2.036	1.921
5	-	2.659	2.336

Summary:

$$i = \max\left(1.209\ ,\ 0.488 + 0.306\left(\frac{d}{t}\right) + 0.0127\left(\frac{d}{t}\right)^2\right) \qquad (6)$$

The strength of the competition strut P_{comp} is the least of the values of P_G, P_C and P_S, where

$$P_G = 1.85\frac{E}{i}\frac{d^2D^2}{L^2} \qquad (7)$$

$$P_C = 1.79\frac{E}{i}\frac{n^2d^4}{L^2} \qquad (8)$$

$$P_S = 2.36\frac{E}{1.209}(0.943t)^2\sin^2\theta\cos\theta$$
$$\text{where}\quad \theta = \tan^{-1}0.866\frac{nD}{L} \qquad (9)$$

The strength of the reference strut is

$$P_{ref} = 0.785E\frac{(V_B+V_C)^2}{L^4}$$

$$\text{where}\quad V_B = \frac{1.5}{1.209}(n-2)\pi(0.943t)^2\sqrt{0.375D^2+\frac{L^2}{n^2}}$$

$$\text{and}\quad V_C = 2.36\frac{Ld^2}{i}\left(1+\frac{0.785D^2}{L^2}\right) \qquad (10)$$

The score is P_{comp}/P_{ref} (11)

Table 4: Optimal 'Strut 2' performance with decreasing minimum thickness with triangular bracing and chords with triform sections

Minimum thickness t (mm)	No. of segments n	Pitch circle dia. D (mm)	Virtual chord diameter d (mm)	Relative stiffness factor i	Volume V (cc)	Strength P (N)	Specific strength P/V (N/cc)	Score
2	11	52.0	4.81	1.297	13.44	3685	274	25.2
1.75	11	49.7	4.60	1.379	10.97	2891	264	29.7
1.5	12	48.0	4.07	1.412	8.43	2065	245	35.9
1.25	13	45.9	3.59	1.471	6.21	1404	226	45.0
1	14	43.2	3.14	1.572	4.31	890	206	59.2
0.75	17	40.6	2.43	1.611	2.59	459	178	84.9
0.5	21	36.8	1.78	1.739	1.29	188	146	140.5

Table 5: Geometries of 4 optimum latticed struts

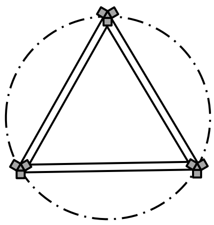
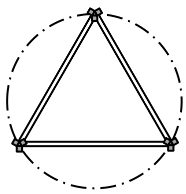
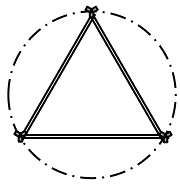
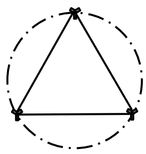
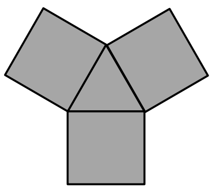
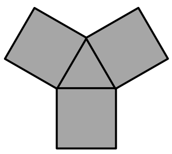
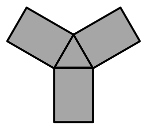
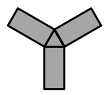
Minimum thickness	2 mm	1.5 mm	1 mm	0.5 mm
Central cross section Each to the same scale				
Chord cross section Each to the same scale				
Pitch circle dia.	52.0 mm	48.0 mm	43.2 mm	36.8 mm
Factor, i	1.297	1.412	1.572	1.739
Fin length, f	2.05 mm	1.84 mm	1.50 mm	0.88 mm
Failure strain	5.31%	4.53%	3.66%	2.66%
Score	25.2	35.9	59.2	140.5

Table 4 shows the result of the optimization based on Eqns 6-11. Table 5 presents the geometry defined by the values of t, d and i, also the strains at which buckling occurs.

These 'optimal' latticed struts, constrained only by material thickness, buckle in the three modes, global buckling, local chord buckling and local shear buckling simultaneously.

Changing n, D or d from the tabulated values reduces both the score and the specific strength of these struts, which is a reasonable claim to optimality.

Buckling strain is a principal design parameter. The three buckling strains for this strut configuration are:

$$\varepsilon_G = 0.785\frac{D^2}{L^2} \qquad \text{Global buckling}$$

$$\varepsilon_C = 0.762\frac{n^2 d^2}{L^2} \qquad \text{Local chord buckling}$$

$$\varepsilon_S = 0.737 i \frac{t^2}{d^2}\sin^2\theta\cos\theta \qquad \text{Local shear buckling}$$

where $\theta = \tan^{-1} 0.866\frac{nD}{L}$

and $i = \max\left(1.209\ ,\ 0.488 + 0.306\left(\frac{d}{t}\right) + 0.0127\left(\frac{d}{t}\right)^2\right)$

Each buckling strain is dictated purely by shape, by proportion. Young's modulus has no effect on buckling strains, which are determined solely by the number n and the ratios $D : d : t : L$. Whereas an 'optimal' strut has been defined as one having all buckling strains equal, responsible design will involve choosing the ratios $\varepsilon_G : \varepsilon_C : \varepsilon_S$ as well.

Torsional buckling

When the minimum thickness is reduced below 0.5 mm another buckling mode becomes significant, the torsional buckling mode of the triform section. Fig 14 shows a slender triform section having fins of length, l, measured from the centreline, and thickness, t, buckling under an axial load, P, into a spiral shape, each edge lying on a helix. If the twist per unit length in the axial direction is θ, then, at a point A, r from the member's axis, the inclination of the surface to the axial direction is $\phi = r\theta$. The force on a t by δr wide element at A is $\delta P = \frac{Pt\delta r}{3lt}$, if the load, P, is distributed uniformly across the section. The tangential component of this force is $\phi\delta P = r\theta\delta P$. Integrating this force multiplied by its lever arm, r, over the whole section, gives the torque, T, experienced by the whole section as

$$T = 3\int_0^l r^2\theta\frac{P\mathrm{d}r}{3l} = \frac{1}{3}Pl^2\theta$$

The deformation caused by this torque is

$$T = GK\theta$$

where G is the material's shear modulus and K is the section's torsional constant, which, considering the section to be equivalent to three plates, l wide and t thick, is, from ref. [3]

$$K = lt^3\left[1 - 0.63\frac{t}{l}\left(1 - \frac{t^4}{12l^4}\right)\right]$$

Equating the two expressions for the torque, T,

$$\frac{1}{3}Pl^2\theta = Glt^3\left[1 - 0.63\frac{t}{l}\left(1 - \frac{t^4}{12l^4}\right)\right]\theta$$

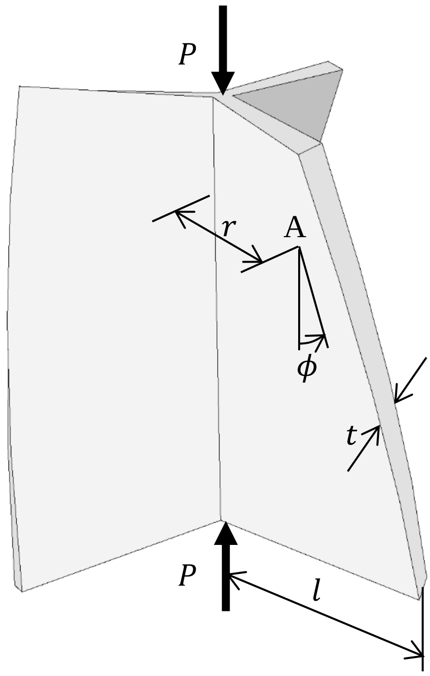


Figure 14: Torsional buckling of a triform section

Dividing P by the sectional area, $3lt$, and Young's modulus, $E = 2G(1+\nu)$, the torsional buckling strain is

$$\varepsilon_T = \frac{1}{2(1+\nu)}\frac{t^2}{l^2}\left[1 - 0.63\frac{t}{l}\left(1 - \frac{t^4}{12l^4}\right)\right]$$

where ν is the Poisson's ratio of the material, typically about 0.3.

When $l/t = 10$, its torsional buckling strain is 0.36%. The triform chords of the 0.5 mm minimum thickness optimal strut have an external fin length of 0.88 mm, so $l/t = 2.04$, when account is taken of the additional 0.14 mm to the member's centreline. Its torsional buckling strain is 6.4%, well above its 2.66% failure strain, so torsional buckling is not an issue for the struts of Table 5.

The torsional buckling strain, ε_T, is independent of the length of the member, so it is a section property.

Triform competition struts

Each fin of a triform competition strut has the minimum thickness, t. The fin length, l, from the member's centreline, is the only free variable, which must be optimised to give the maximum score. There is no reason why the sectional area of this strut should be constant along its length. Instead, a flexural stiffness variation proportional to the square of the profile function, p, should dictate how the section varies along its length if it is to be analysed using the properties of the standard tapered strut. In practice, giving the tip of each fin this profile is sufficiently accurate.

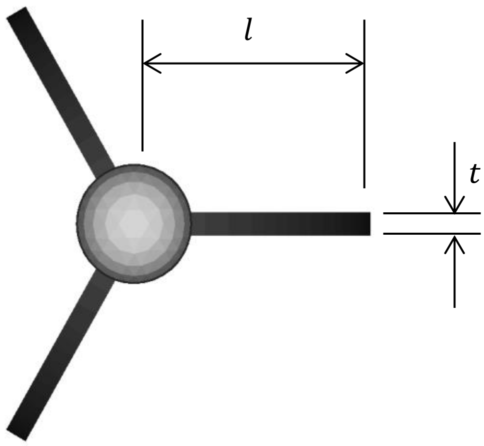


Figure 15: End-on view of a triform competition strut

The sectional area and inertia at the middle section are given by

$$A_0 = 3l_0 t$$

$$I_0 = \frac{3}{2}\left(\frac{t{l_0}^3}{3} + \frac{t^3 l_0}{12}\right) = \frac{l_0 t\left(4{l_0}^2 + t^2\right)}{8}$$

The global buckling strength of the strut is

$$P_G = \frac{2\pi E I_0}{L^2}$$

The torsional buckling strength of the strut is

$$P_T = EA_0\varepsilon_T = \frac{EA_0}{2(1+\nu)}\frac{t^2}{{l_0}^2}\left[1 - 0.63\frac{t}{l_0}\left(1 - \frac{t^4}{12{l_0}^4}\right)\right]$$

If the area of the strut were to follow the profile function, p, then the volume of the strut would be $AL/\sqrt{2}$. This gives the reference strut's diameter, d_{ref}, as

$$ {d_{ref}}^2 = \frac{2\sqrt{2}}{\pi}A$$

and $$P_{ref} = \frac{\pi^3 E {d_{ref}}^4}{64L^2} = \frac{\pi E A^2}{8L^2}$$

Fig 16 shows plots of strength versus fin length, l, for when the thickness, $t = 1.5$ mm and 1 mm respectively. In each plot the strength of the competition strut is the lesser of the two buckling strengths, and the score is the ratio of this to the strength of the reference strut shown by the bottom curve. In both cases, the optimal strut is defined by the intersection of the two upper curves, when $l = 13.39$ mm and 10.95 mm, and the associated scores

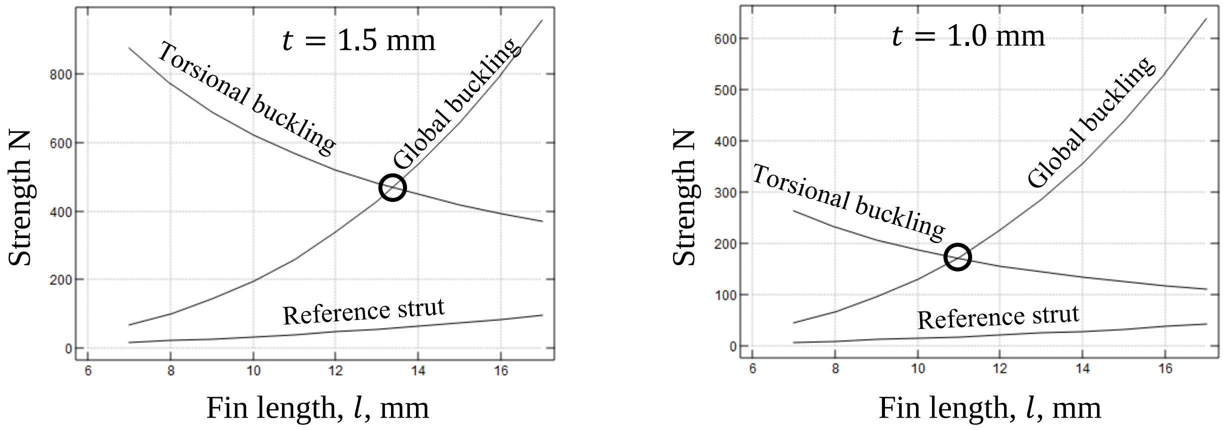


Figure 16: Strengths of triform struts of thickness 1.5 mm and 1 mm

are 7.96 and 9.75. The strains at the middle section are 0.471% and 0.315%, both uncompetitive with the Strut 2 topology.

The profile of the fins needs to be adjusted close to the ends of a triform strut to avoid material overstress. Fig 17 shows the default profile joining the end sphere on the left-hand side and, next to it, a suitably modified profile.

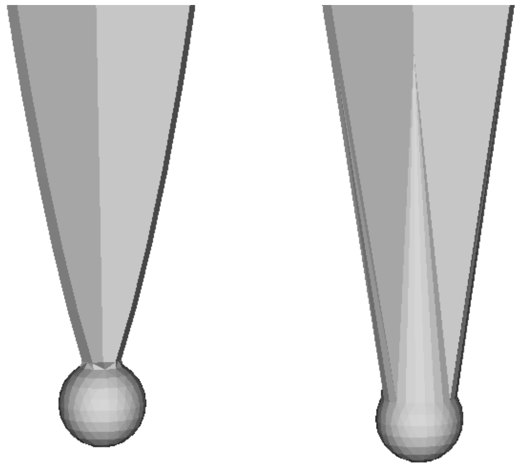


Figure 17: Modification at the ends of a triform strut needed to avoid material overstress

Conclusion

The competition strut is an enduring challenge to engineering ingenuity. The design brief is simple, with no limits to loading, support, or boundaries, except that it should be 200 mm long. It has been shown how the score is limited by material thickness: smaller thicknesses result in higher scores, so a material and a manufacturing process will be chosen that can accurately produce quality material as thinly as possible. This choice defines a minimum thickness as the only design constraint. Neither the stiffness nor the density of the material affects the score. The issue is shape.

If the requirement for the testing of physical specimens is waived, scores may be calculated using buckling analysis software, when designs for struts with thicknesses too small to be manufactured will produce much higher scores. Another article will explore how to proceed when the thickness constraint is removed. The issue, then, is only shape, and the higher the score the more intricate that shape must be.

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3. R. J. Roark, W. C. Young, Formulas for Stress and Strain, 5th Edn., McGraw-Hill 1976, p. 290.